TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 5 2019

Section 1 Calculator Free

Differential Equations, Implicit Differentiation, Related Rates

STUDENT'S NAME SOLUTIONS

DATE: Monday 19 August **TIME:** 20 minutes **MARKS:** 20

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (4 marks)

Determine the general solution for the differential equation $\frac{dy}{dx} = x^2y^2 - 2x^2y + x^2$.

$$\frac{dy}{dx} = x^2(y^2 - 2y + 1)$$

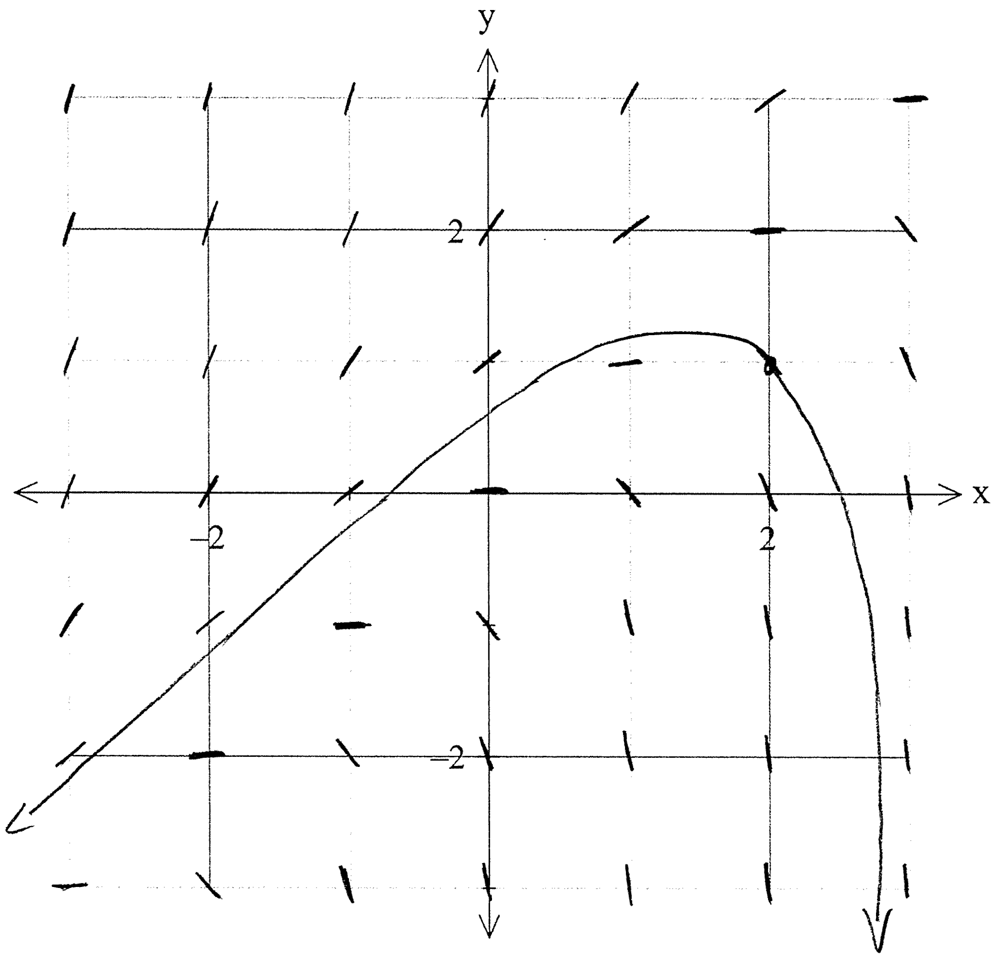
$$\int \frac{dy}{(y-1)^2} = \int x^2\,dx$$

$$\int (y-1)^{-2}\,dy = \int x^2\,dx$$

$$\frac{-1}{y-1} = \frac{x^3}{3} + c$$

2. (5 marks)

(a) On the axes below sketch the slope field for the differential equation $y' = y - x$.
Use integer values of x and y only. [3]



(b) Sketch the particular solution to the differential equation on the axes above that passes through the point (2,1). [2]

3. (4 marks)

Determine an expression for $\frac{dy}{dx}$ given $y^2 - ye^{\cos x} + 2x = \tan\frac{2\pi}{3}$

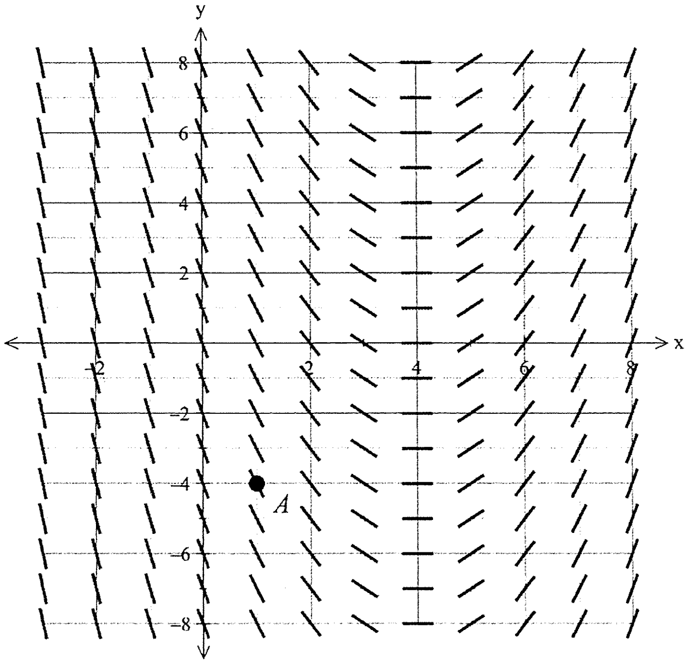
$$2y.y' - (e^{\cos x}.y' - y\sin x e^{\cos x}) + 2 = 0$$

$$y'(2y - e^{\cos x}) = -2 - y\sin x e^{\cos x}$$

$$y' = \frac{-2 - y\sin x e^{\cos x}}{2y - e^{\cos x}}$$

4. (7 marks)

A first order differential equation has a slope field as shown below.



(a) Give a reason why the general solution of the differential equation for the slope field is quadratic and a reason why the differential equation is linear. [2]

D.E. HAS ONT T.P. ∴ ORIGINAL FUNCTION QUADRATC

DERIVATIVE OF A QUADRATIC IS LINEAR

The slope field at the point $A(1,-4)$ has a value of -3.

(b) Determine the equation of the curve containing point A. [5]

$y = ax^2 + bx + c$

$y' = 2ax + b$

$x = 4$, $y' = 0$ $\qquad 0 = 8a + b$

$x = 1$, $y' = -3$ $\qquad -3 = 2a + b$

$3 = 6a$

$\frac{1}{2} = a$

$-4 = b$

$y = ax^2 + bx + c$

$y = \frac{x^2}{2} - 4x + c$

$(1,-4)$ $\quad -4 = \frac{1}{2} - 4 + c$

$c = -\frac{1}{2}$

$y = \frac{x^2}{2} - 4x - \frac{1}{2}$

TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 5 2019

Section 1 Calculator Assumed
Differential Equations, Implicit Differentiation, Related Rates

STUDENT'S NAME ______________________________

DATE: Monday 19 August **TIME:** 30 minutes **MARKS:** 30

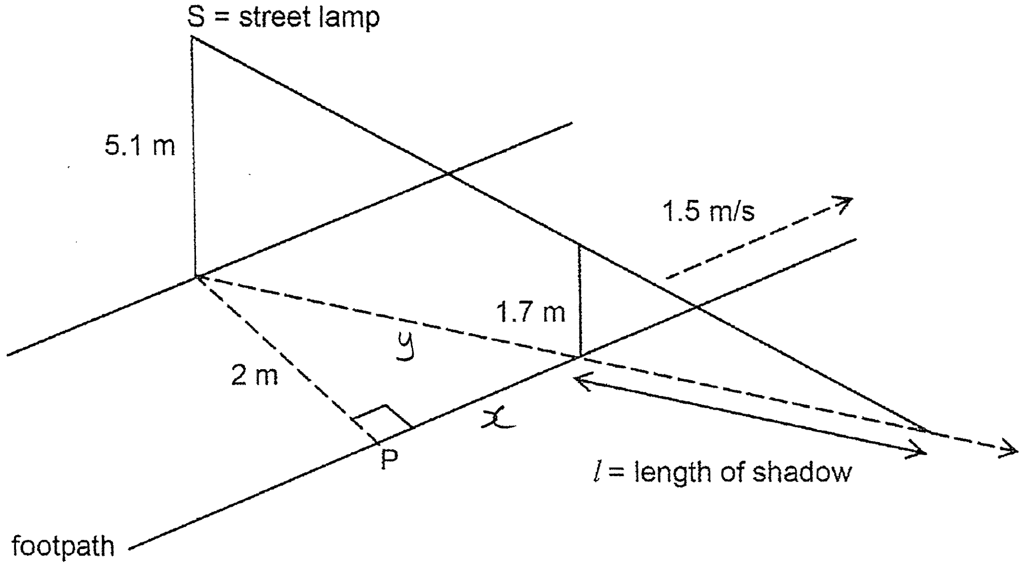
INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

Intentional blank page

5. (6 marks)



In the diagram above, P is the initial position of a boy, of height 1.7 metres, who is walking along a straight footpath in the direction shown.

S is the position of a street lamp of height 5.1 metres with its base 2 metres from P.

The street lamp will cast a moving shadow of the boy as he continues to walk along the footpath at 1.5 m/s.

(a) If x metres is the distance walked by the boy from P, show that the length, l metres, of the boy's shadow is given by $l = \frac{1}{2}\sqrt{4+x^2}$. [3]

$$y = \sqrt{4+x^2}$$

$$\frac{l}{1.7} = \frac{\sqrt{4+x^2}+l}{5.1}$$

$$l = \frac{\sqrt{4+x^2}+l}{3}$$

$$3l = \sqrt{4+x^2}+l$$

$$2l = \sqrt{4+x^2}$$

$$l = \frac{\sqrt{4+x^2}}{2}$$

(b) Determine the position of the boy when the rate of change of the length of the boy's shadow is 0.25 m/s. [3]

$$l = \frac{\sqrt{4+x^2}}{2}$$

$$\frac{dl}{dt} = \frac{\frac{1}{2}(4+x^2)^{-\frac{1}{2}} \cdot \cancel{2}x}{\cancel{2}} \quad \frac{dx}{dt}$$

$$\frac{1}{4} = \frac{x}{2(4+x^2)^{\frac{1}{2}}} \cdot \frac{3}{2}$$

$$\frac{1}{3} = \frac{x}{(4+x^2)^{\frac{1}{2}}}$$

$$(4+x^2)^{\frac{1}{2}} = 3x$$

$$4+x^2 = 9x^2$$

$$4 = 8x^2$$

$$\frac{1}{\sqrt{2}} = x$$

6. (9 marks)

The rate with which $\theta° C$, the air temperature at altitude h metres, decreases with respect to altitude, is directly proportional to the sum between the air temperature and at altitude h and 273. The temperature at sea level, altitude = 0 metres, is assumed to be $20°C$.

This relationship is given by $\frac{d\theta}{dh} = -k(\theta + 273)$ where k is a constant.

(a) Use calculus to determine an equation for θ in terms of h. [5]

$$\int \frac{d\theta}{\theta + 273} = \int -k\,dh$$

$$\ln(\theta + 273) = -kh + c$$

$$\theta + 273 = Ae^{-kh}$$

$t = 0$

$\theta = 20$

$$293 = A$$

$$\theta = 293e^{-kh} - 273$$

Given $k = 3 \times 10^{-5}$;

(b) Calculate the air temperature at 3 km. [1]

$h = 3000$ $\theta = -5.2°C$

(c) Determine the height at which the air temperature is $-60°C$. [1]

$\theta = -60°$ $h = 10629.3\,m$

(d) Determine the rate at which the temperature is changing when the height is 2 km. [2]

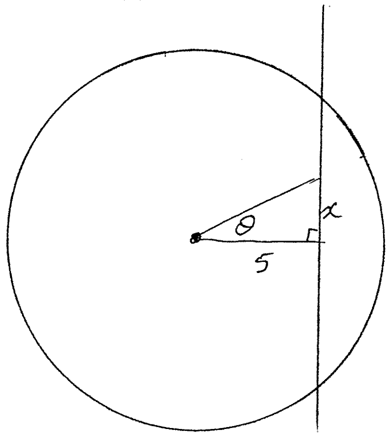
$$\theta = 293e^{-0.00003 \times 2000} - 273$$
$$= 2.937$$

$$\frac{d\theta}{dh} = 0.00003(2.937 + 273)$$
$$= 0.00828\,°C/m$$

7. (7 marks)

A rotary sprinkler sprays a single jet of water out from its centre and rotates clockwise on its base at a speed of 4 revolutions per minute. The sprinkler's water jet reaches a maximum distance of 10 metres. The sprinkler is situated 5 metres away from the nearest point P on a straight wall.

(a) How fast is the jet of water moving along the wall when it is 80 cm from P. [4]

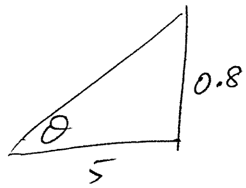


$\tan\theta = \frac{x}{5}$

$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\frac{1}{\cos^2 0.159} \times 8\pi = \frac{1}{5}\frac{dx}{dt}$

$128.9\ m/min = \frac{dx}{dt}$



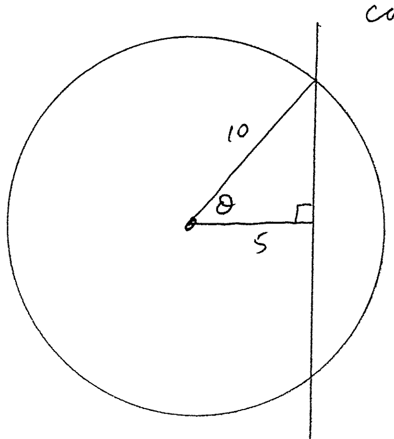
$\tan\theta = \frac{0.8}{5}$

$\theta = 0.159$

$\frac{d\theta}{dt} = 8\pi\ rad/min$

(b) What is the fastest speed the water jet reaches when moving across the wall and where does this occur? [3]

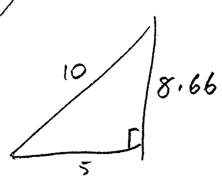
$\cos\theta = \frac{5}{10}$



$\frac{1}{\left(\frac{5}{10}\right)^2} \cdot 8\pi = \frac{1}{5}\frac{dx}{dt}$

$502.7\ m/min = \frac{dx}{dt}$

OCCURS $x = \pm 8.66\ m$ FROM NEAREST POINT SPRINKLER IS TO THE WALL.



8. (8 marks)

With the removal of natural predators, the population of a species of marsupial on a nature reserve is expected to follow a logistic growth model given by $\frac{dP}{dt} = \frac{P}{4} - \frac{P^2}{2500}$ for t years.

(a) Determine an expression for P in terms of t in the form $P = \frac{K}{1 + Ce^{-at}}$ if there is an initial population of 80 animals. [4]

$ax - bx^2$

$a = \frac{1}{4}$ $b = \frac{1}{2500}$

$$P = \frac{\frac{a}{b}}{1 + ce^{-at}}$$

$$P = \frac{625}{1 + Ce^{-0.25t}}$$

$t = 0$
$P = 80$

$$80 = \frac{625}{1 + c}$$

$$c = 6.8125$$

$$P = \frac{625}{1 + 6.1825e^{-0.25t}}$$

(b) How long will it take to reach half of the limiting population? [2]

$$312.5 = \frac{625}{1 + 6.1825e^{-0.25t}}$$

$t = 7.7$ YRS

(c) For what population is the growth rate quickest? [2]

$$\frac{d^2P}{dt^2} = \frac{1}{4} - \frac{2P}{2500}$$

$$\frac{1}{4} - \frac{2P}{2500} = 0$$

$$P = 312.5$$

312 OR 313